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Partial stability of catastrophe sections and its application to the cuspoids and conic umbilics

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Abstract. The control space of a system described by an elementary catastrophe frequently has (locally at least) a cartesian product structure, caused for example by physical distinctions among control variables or by observational necessity. The effect of this structure, like that of symmetries, can be to stabilise singular catastrophe sections (partial unfoldings). We define 'partial stability' to be the structural stability of a stratum of a catastrophe section under perturbations respecting the product structure. These ideas are illustrated by reference to a recent observation of 'double-cusp diffraction', and then applied to our previous studies of singular catastrophe sections.

1. Introduction

Elementary catastrophe theory was introduced by Thom (1972), extended by Arnol'd (1975) and explained simply by Poston and Stewart (1978) and Gilmore (1981). For a guide to its applications see Zeeman (1977), Poston and Stewart (1978), Berry and Upstill (1980), Stewart (1981, 1982). Zeeman (1982) sets it clearly within the context of dynamical systems: it provides a general model for systems governed by a gradient dynamic.

An elementary catastrophe is generated by a function $\phi: S\times C\rightarrow \mathbb{R}$, where S is the n-dimensional *state space* manifold and C is the K-dimensional *control space* manifold. The theory describes for a typical such ϕ the local behaviour of its *critical points* with respect to $s\in S$ (points where $d_s\phi(s, c)=0$) as $c\in C$ is varied. Of particular importance is the *bifurcation set* $\mathscr{B}\subset C$ on which the critical points of ϕ are degenerate ($d_s^2\phi(s, c)$ singular): as $\mathscr{B}$ is crossed the critical point structure of ϕ changes. This leads to striking physical changes (catastrophes) in a system which is modelled so that critical points of ϕ correspond to equilibrium states.

Because the theory is local, S and C may be replaced by their tangent spaces $\mathbb{R}^n$ and $\mathbb{R}^K$ respectively, so in this paper we will identify S with $\mathbb{R}^n$ and C with $\mathbb{R}^K$; indeed, in applications this identification is often globally valid. We will assume that the main singularity of ϕ occurs at the origin of $S\times C$. Two functions $\phi, \psi: S\times C\rightarrow\mathbb{R}$ are regarded as equivalent if one can be made equal to the other by a smooth diffeomorphism of $S\times C$ which preserves its form as a fibre bundle over C, and an elementary catastrophe is an equivalence class of such functions that is stable to small

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perturbations (although modal catastrophes such as X_9 discussed below are a little more complicated). A catastrophe is conveniently represented by a generating function that is a simple polynomial *normal form* (that these exist is one of the profound results of the theory). It is perhaps a surprising empirical result that generating functions arising directly from physical models are frequently very close locally to the canonical normal forms arising from the mathematics (see § 4 for an example). Consequently, we shall mainly assume that we are dealing with a normal form.

For the equivalence (and hence the notion of stability) described above to make sense there should be no additional structure in $S \times C$, but in practice there frequently is. This paper is concerned with the existence of distinguished directions or subspaces in C which give it (locally) a product structure of the form

$$C \equiv \mathbb{R}^{K_1} \times \mathbb{R}^{K_2} \times \ldots \times \mathbb{R}^{K_m}, \qquad m \leqslant K, \qquad \sum_{i=0}^{m} K_i = K.$$

Then only diffeomorphisms which preserve this structure strictly produce equivalent systems. Nevertheless, we take the view that it can be useful to regard this structure as imposed *a posteriori* onto a catastrophe existing (stably) in a structureless C. This is consistent with many existing applications of elementary catastrophe theory in which any structure in C has been completely ignored. However, it contrasts with the point of view taken by Wassermann (1975), who has classified unfoldings that are stable under an equivalence respecting *a priori* a fibration of the form $C \equiv \mathbb{R}^3 \times \mathbb{R}$. The idea of distinguished control variables also underlies Golubitsky and Schaeffer's (1979a) 'theory for imperfect bifurcations via singularity theory' (included in Stewart's (1981, 1982) reviews). These approaches have been generalised by Damon (1983), who has developed a general unfolding theory that accounts for such factorings, but without giving any classification results.

In § 2 we examine the origins of factored control spaces and show how they lead to catastrophe sections and partial unfoldings. Our particular concern is the stability of such partial unfoldings apart from the main singularity, under perturbations that respect the product structure of C. We develop the notion of *partial stability*, for which we establish conditions by using transversality in § 3. Section 4 illustrates the ideas by applying them to a recent experimental and theoretical investigation of 'double-cusp diffraction' by Upstill *et al* (1982). In § 5 we achieve our main purpose, which is to use the methods that we have established to enhance our previous studies of the infinite sequences of cuspoid and conic umbilic catastrophes via their singular coordinate sections (Wright (1981) and Wright *et al* (1982), respectively) and illustrate how this new information may be used. The technical details of the stability calculations are described in an appendix for the case of the conic umbilic sections, which is the most involved example that we have considered and hence subsumes the others. They primarily involve reducing the local unfolding of a singularity to normal form, of which there do not seem to be many detailed examples in the literature.

2. Catastrophe sections and partial unfoldings

Typically, a high-codimension catastrophe is observed not within the whole of C at once, but within some family of affine subspaces of C; in other words, it is observed via a family of sections. By a catastrophe section $\phi_{\mathscr{C}}$ we mean the generating function ϕ restricted to $\mathscr{C} \subset C$, i.e. $\phi|_{S \times \mathscr{C}}$, which for many purposes is adequately described by specifying the diffeotype of ϕ within each region (stratum) of $\mathscr{C}$ where it is constant. If the origin $0 \in C$, at which the main singularity of ϕ occurs, lies in $\mathscr{C}$, then $\phi_{\mathscr{C}}$ is a *singular section* of ϕ, i.e. a *partial unfolding* of $\phi|_{S\times\{0\}}$. Only singular sections capture the intrinsic structure of the main singularity; non-singular sections display only the global structure organised by it. The subspace of C within which a catastrophe is actually observed constitutes a factor of C, but this may itself have a product structure, as may the remaining factor of C; furthermore, the factors of C are frequently spanned by the axes used to define normal forms. These properties will be illustrated in detail in § 4. We now describe two ways in which distinguished directions, or product structures, in C arise.

Firstly, in modelling for example physical systems, the control variables generally do not all represent the same physical quantities: they might represent such diverse parameters as space, time, temperature, pressure, etc. It is probably reasonable to regard as physically equivalent two systems that are mathematically equivalent under a transformation among variables of only one type, such as among only spatial variables, or among only pressure variables, but if the necessary transformation mixes up for example space and pressure then the two systems may not be at all similar physically. Of course, the state variables may also represent physically distinct quantities, but we will not consider the effects of this here. Furthermore, some potential control parameters may be much more readily accessible experimentally than others, so that a catastrophe is naturally observed within families of affine subspaces of the control space, spanned by the most accessible controls, rather than within the whole control space at once. For example, an optical caustic (which corresponds to $\mathscr{B}$) in three dimensions is most easily observed by photographing two-dimensional sections of it. One has some choice about the orientation of the sections, but it is most natural to take them perpendicular to the dominant ray direction, which is once again a physical distinction among directions in control space. In fact, Nye and Hannay (1984) show that only certain caustic orientations are possible, a concept that has meaning only in the presence of such distinguished directions.

It is quite possible that only a singular section of a catastrophe is readily accessible in some particular experiment, despite the fact that it is *a priori* unstable. For example, such a section that displays a particular symmetry may in fact be stable if the experiment imposes this symmetry. The question: 'What happens if the symmetry is relaxed?' is at least partly answered by examining the partial stability of the unstable symmetric section, in that it shows what happens when new unfolding parameters that break the symmetry become available. Symmetry breaking has received considerable attention recently (e.g. Golubitsky and Schaeffer (1979b), see also Stewart's (1981, 1982) reviews), but at present only scattered results are available on equivariant catastrophes (e.g. Poènaru 1976, Wassermann 1977, Damon 1983).

Sectioning of a catastrophe is more important the higher is its codimension K: for example, the codimension-8 (unimodal) catastrophe X_9 (Arnol'd 1973b, Godwin 1975, Callahan 1982) is important in constructing global models because among compact catastrophes (such that ϕ always has a global extremum at finite s) of corank 2 it has the lowest codimension. A general outline of the significance of X_9 for mathematics and applications is given by Zeeman (1976b, see also Zeeman 1977). It has also been applied, for example, to von Karman buckling of thin elastic plates (Chow *et al* 1975, 1976, List 1977, Magnus and Poston 1977, Poston and Stewart 1978, pp 317–24) and to phase transitions (Keller *et al* 1979). There is evidence that this and higher catastrophes are particularly important in optics (Berry 1977, Upstill 1979a, b, Nye 1979, Berry and Upstill 1980, Berry 1982, Walker *et al* 1983). In view of its importance, we shall return to X_9 in § 4 as an illustration of the above remarks (in an optical setting).

Secondly, distinguished control directions arise artificially in studies of the geometry of higher catastrophes (of codimension greater than 3). Such studies frequently proceed by examining the catastrophe within families of two- (or possible three)-dimensional affine subspaces of C, simply because of the difficulty of presenting (not to mention comprehending) information in more dimensions. See particularly Callahan (1977), where the name *tableau* is introduced for a complete family of catastrophe sections; for further examples see Godwin (1971), Woodcock and Poston (1974), Upstill (1979a), Nye and Thorndike (1980), Callahan (1982, 1980, 1981). For catastrophes of high codimension it is possible to present only certain (arbitrary) subfamilies of sections. Wright (1981) and Wright *et al* (1982) have studied the general cuspoid and conic umbilic catastrophe in a particular set of mutually orthogonal plane sections through the origin of C, called singular coordinate sections, which have all but two of the control variables in a canonical normal form set to zero. These sections illustrate, among other things, some of the family resemblances among the sequences of related catastrophes, but they do not directly give any information about the rest of C. However, by supplementing such sections with information about their partial stability to translations we recover some information about a neighbourhood in C of the plane of section. It was this application that initially motivated the analysis described in the present paper.

3. Partial stability

An important property of a catastrophe section is its structural stability, i.e. whether it stays in the same equivalence class under a small perturbation. Like the whole catastrophe, a generic section is structurally stable, but familiar examples of unstable two-dimensional catastrophe sections are the beak-to-beak and lips events (e.g. Berry and Upstill 1980). The most interesting sections are, of course, the unstable ones, and the most unstable sections are those containing singularities of the highest codimension; in our case 'singular sections' through the origin of C.

Because the whole catastrophe is stable, the effect on $\phi_{\mathscr{C}}$ of perturbing ϕ must be locally equivalent to that of varying the control parameters in an open neighbourhood of $\mathscr{C}$, so we will consider explicitly only the latter form of perturbation. Writing $C \equiv P \times Q$, $q \in Q$, and taking $\mathscr{C} \equiv P \times \{q\}$, we ask whether $\phi_{P\times\{q\}}$ is stable to small variations of q. It suffices to consider the stability of ϕ on $\mathscr{B} \cap \mathscr{C}$, the bifurcation set in $\mathscr{C}$, because for $c \notin \mathscr{B}$, ϕ has codimension 0 and is hence completely stable.

In considering the stability of a catastrophe section we respect the product structure of C by considering perturbations involving variations of q within each of the distinct factors to be *independent*. We will call this *partial stability* with respect to perturbation within a specific factor of C. The stratification of C (into smooth submanifolds—see Bröcker and Lander (1975), Zeeman (1976a, b), Lu (1976)) induces a stratification of $\mathscr{C}$, and it is both most convenient and most informative to analyse the partial stability of the strata individually. Indeed, for a singular section this is essential, because we know *a priori* that the whole section is unstable.

Let $\mathscr{S}^{K-m}$ represent a stratum of codimension m in C, and let us call its intersection $\mathscr{S}^{K-m} \cap \mathscr{C}$ with some 'observation subspace' $\mathscr{C}$ of C a *feature* of $\phi_{\mathscr{C}}$. Let us now denote the distinct factors of Q by $\{Q_i\}$, so that $C \equiv P \times Q_1 \times \ldots \times Q_j$. We define a feature to be *partially stable* with respect to Q_i if its diffeotype as a point set is unchanged by variations of $c \in C$ lying entirely within Q_i, i.e. variation only of the Q_i-components of c.

To analyse partial stability, we recall that two manifolds intersect transversally, and hence stably, if their tangent spaces at each point of the intersection span the ambient space (Gilmore 1981, ch 22, Chillingworth 1977, Poston and Stewart 1978, p 63). Hence a feature $\mathcal{S}^{K-m} \cap \mathscr{C}$ will be partially stable with respect to Q_i if

$$T_c\mathcal{S}^{K-m} \oplus T_c\mathscr{C} \supset T_{c_i}Q_i$$

for all $c \in \mathcal{S}^{K-m} \cap \mathscr{C}$, where c_i is the projection of c onto Q_i. Here we are regarding the 'perturbation space' Q_i as the ambient space.

The tangent space $T_c\mathcal{S}^{K-m}$ at some point c of the stratum $\mathcal{S}^{K-m}$ can be determined from the unfolding of the local singularity by reducing it to a normal form. Clearly, one need work only to linear order locally in the control variables. Methods for finding the normal form to which an unfolding may be reduced are outlined by Poston and Stewart (1978) and Gilmore (1981). (This is a much easier problem than actually finding the transformation that exactly accomplishes the reduction.) As an example, and to support our discussion in § 5, we perform explicitly in the appendix the calculations for the general conic umbilic catastrophe.

Partial stability of a feature is most readily determined by inspecting the set of m linear equations defining $T_c\mathcal{S}^{K-m}$: the feature is unstable to any change in controls that violates these equations in a way that cannot be compensated by a variation of c within $\mathscr{C}$. In our applications in §§ 4 and 5, features corresponding to the intersection of two strata will be important—for further discussion of such intersection strata see Callahan (1982). The set of equations defining the tangent space of an intersection stratum is the union of the equations defining the tangent spaces of each of the intersecting strata, as used below and illustrated at the end of the appendix.

Let us consider in a little more detail the case of two-dimensional sections, where $\mathscr{C}$ (and hence P) is a plane. To be specific, and because it is the approach we will take for our analysis of cuspoid and conic umbilic catastrophes, let us regard the rest of C to be maximally factored as

$$C \equiv P \times \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{(K-2)\text{ times}}, \qquad c \equiv ((a, b), c_1, c_2, \ldots, c_{K-2}) \in C.$$

Then the partial stability of a feature of $\mathscr{C}$ is its stability under translation of $\mathscr{C}$ along any one of the axes of C not contained in $\mathscr{C}$.

We call a feature *completely stable* if it is partially stable to *all* c_i, which means that $\mathcal{S}^{K-m}$ and $\mathscr{C}$ intersect transversely in $C \equiv \mathbb{R}^K$. If so, they intersect in a manifold of dimension $2-m$. (Add codimensions in $\mathbb{R}^K$.) Hence, as one expects, completely stable two-dimensional features can result only from strata of codimension 0, which are non-singular (Morse); completely stable one-dimensional features can result only from strata of codimension 1, i.e. A_2 (fold) strata; and completely stable point features can result only from A_3 (cusp) strata. (We use Arnol'd's (1973a) notation for singularity types.)

Most features are not completely stable because generally $T_c\mathcal{S}^{K-m}$ and $\mathscr{C}$ do not together span C. This is the case for beak-to-beak and lips events in $\mathbb{R}^3$, at which the rib (cusped edge of $\mathscr{B}$)—a codimension 2 stratum $\mathcal{S}^1$—is *tangent* to the plane of section $\mathscr{C}$.

4. Application I: an observation of the $^0X_9$ diffraction catastrophe

To illustrate our theory in the context of modelling an experimental system, we apply it to 'the double-cusp unfolding of the $^0X_9$ diffraction catastrophe' recently studied by Upstill *et al* (1982), henceforth referred to as U82. In their experiment, a pair of orthogonal plane water waves of small amplitude was generated on the water surface of a ripple tank to provide a dynamic refracting interface, and laser light shone vertically through the tank was photographed using a shutter synchronised with the wave generators to act as a stroboscope. The dominant focusing arises from the local summits of the water surface, and a simple mathematical model shows that the resulting singularity in the optical eikonal is the binary quartic denoted $^0X_9$ (Arnol'd 1973b, Callahan 1982), for which a convenient universal unfolding is

$$\phi = x^4 + Kx^2y^2 + y^4 + Ax^2y + Bxy^2 + C(x^2+y^2) + D(x^2-y^2) + Exy + Ux + Vy, \qquad K > -2,\ K \neq 2. \tag{1}$$

This unfolding is eight-dimensional, and it is not surprising that the unfolding parameters were not all readily accessible. We shall consider the correspondence between the canonical control variables in (1) and physical variables only near the main $^0X_9$ focus. Then in order of decreasing accessibility: U and V correspond to spatial directions perpendicular to the laser beam, so that every photograph displays a (U, V) section; C corresponds to the spatial direction parallel to the laser beam, i.e. the focusing height; and D corresponds to the difference in amplitude of the water waves. The physical control variables C and D are related by a 45° rotation to those that would result from taking monomial unfolding terms. The symmetry of the experiment requires that $A = B = E = 0$, and for small amplitude water waves (so that the focal length is very long) K is very small and might be taken as 0 to a first approximation (but see below). With $A = B = E = K = 0$, (1) separates into a sum of two cusp (A_3) unfoldings, which is why this partial unfolding was called 'the double-cusp unfolding of $^0X_9$'.

The double-cusp unfolding is a partial unfolding of $^0X_9$ and is hence highly unstable, so that one might expect it to be very difficult to observe. However, it is stable subject to the symmetries imposed by the experiment. The condition of far-field observation that makes K (approximately) zero also corresponds to a symmetry: essentially translation invariances of the wavefunction. The interesting question is: 'What happens when the symmetries are broken, in other words, how does this partial unfolding fit into the full unfolding of $^0X_9$?' To answer this question we need to know (topologically, at least) how ϕ will change if any one of A, B, E or K is perturbed slightly from zero. We know that the main singularity of the caustic will break up, since it is completely unstable (as discussed in § 3). However, this local break-up will be completely obscured by diffraction for sufficiently small perturbations. Instabilities of features of the double-cusp unfolding having lower codimension are likely to be much more readily observable, because they will cause global changes in the caustic. It is precisely to such instabilities that our partial stability analysis applies.

U82 show that E may be varied from 0 by changing the angle between the water waves from 90° (and hence breaking a symmetry), and perturbing K from zero corresponds to removing the restriction to far-field observation. However, varying A and B, and achieving the full significant range of K values, would probably require adding a third water wave; in other words, rebuilding the apparatus! This and the above discussion of U, V, C and D illustrates clearly how different control parameters can have very different physical significance, and may vary greatly in their accessibility. Experimental inaccessibility of controls stabilises partial unfoldings in practice. Furthermore, the experiment illustrates how physically distinct control parameters tend to correspond to distinct canonical control parameters in a simple way (at least near the main singularity).

The purpose of this section is illustration, for which there is an obvious advantage in restricting dimensions to at most three. So rather than analyse the partial stability of the whole double-cusp unfolding, we will analyse the partial stability of the (U, V, C) singular section of it to individual variations of A, B, E, K, and D away from 0. This follows the approach taken both experimentally and in the analysis of their results by U82. Because (U, V, C) is locally isomorphic to the space in which the physical focus forms, the (abstract) caustic in this experiment is naturally observed as a family of (U, V, C) sections. These could in principle be observed directly by, for example, blowing smoke around the focus and observing the scattered light, although U82 actually observed each (U, V, C) section as a stack of (U, V) sections each at constant C by varying the focusing of the camera (see figures 4(a)–(h) and 9(a)–(g) of U82).

The (U, V, C) singular section is shown in figure 1. Note that the caustic has point symmetry group 4mm. The A_2 (fold) surfaces intersect to give A_2^2 strata along four curves that also coincide with D_4^+ (hyperbolic umbilic) strata, so that the caustic surfaces 'within the cusps' actually consist of three coincident A_2 strata denoted by A_2^3. Hence the double-cusp caustic is actually much more complex than it appears. The origin of this complexity is nicely illustrated in § 4 of Callahan (1982)—see especially figures 19 and 25.

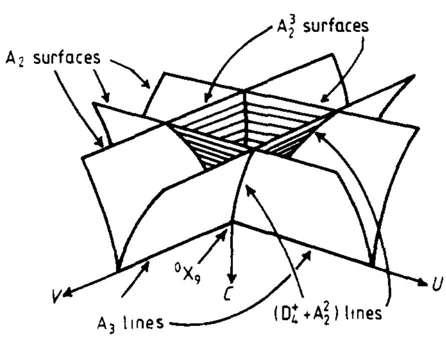

Figure 1. The (U, V, C) singular section of the double-cusp caustic (from figure 1(a) of U82).

Table 1 shows the partial stability of each stratum of the (U, V, C) singular section, although where there are several strata related by the 4mm symmetry, we consider only a single representative. As we remarked in § 3, the tangent space of a stratum is defined by a set of linear equations in the local control variables, the number of equations being equal to the codimension of the stratum. Each parenthesis in the middle column of the table corresponds to one such equation, and shows the control variables appearing in it. If none of U, V or C appears in any parenthesis, then the stratum is unstable to all the controls in that parenthesis, as indicated in the final column of the table. Note that a superscripted stratum has codimension equal to that of the basic singularity multiplied by the superscript, e.g. A_2^3 has codimension 3.

Table 1. Partial stability of features of the singular (U, V, C)-section of ${}^0X_9$.

Singularity type	Sets of local control variables related by tangent space	Stability to A, B, D, E, K of (U, V, C) section
A_2 (surface)	$(A, B, C+D, E, K, U)$	Stable
A_2^2 (line)	$(A, B, C+D, E, K, U)$, $(A, B, C-D, E, K, V)$	Stable
A_2^3 (surface)	(A, E), (B, K), $(C+D, U)$	Stable to D only
A_3 (line)	$(A, C+D, K)$, (B, E, U)	Stable
D_4^+ (line)	(A, B, E, K), $(A, C-D, K, V)$, $(B, C+D, K, U)$	Stable to D only

The essential 'double-cusp' nature of ${}^0X_9$ resides in the structure that is *explicit* in figure 1. This consists of the pair of cusped caustic surfaces, each comprising an A_3 line and two A_2 surfaces, and their intersections in four A_2^2 lines. Table 1 shows that these strata are all completely stable, and will therefore be preserved in all local unfoldings of ${}^0X_9$. In other words, the 'large-scale' structure of the double-cusp unfolding is completely stable. (By contrast, figure 1 makes it clear that all structure of the singular (U, V) section $(C=0)$ is unstable to C.) This observation helps to resolve a paradox remarked upon by U82. It was proved by Callahan (1982) (see also Upstill (1979a)) that the topological type of the ${}^0X_9$ caustic differs for $K <, =, > 0$. The theoretical study of ${}^0X_9$ presented by U82 was for $K=0$, although they showed (in appendix A) that in their experiment K actually had a (very small) positive value. Despite the consequent topological difference they found good agreement between their theory and observations. Our results suggest that this 'experimental stability' to K of double-cusp diffraction does not rely as much on diffraction blurring of the geometrical caustic as was suggested by U82 (p 1670), because although the details of the caustic are unstable, the *dominant structure* is not.

The unstable structure in figure 1 comprises the lines of D_4^+ (hyperbolic umbilic) singularities, which coincide with the A_2^2 lines, and the triple degeneracy of the A_2^3 surfaces that these $(D_4^+ + A_2^2)$ lines organise. This structure is stable only to D, as illustrated in figure 2. The only effect of varying D from zero is to unfold the main $({}^0X_9)$ singularity into a pair of E_6 singularities. By contrast, the instability of the D_4^+ and A_2^3 strata to E is illustrated by the (U, V) section shown in figure 3. (For the corresponding (U, V, C) section see figure 13 of U82.)

In this section we have applied partial stability analysis to a question where, to some extent, we already know the answer, in order to illustrate how it works, and how

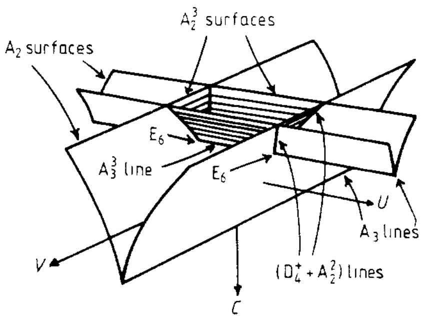

Figure 2. A (U, V, C) section of the double-cusp caustic with $D<0$ (from figure 1(b) of U82).

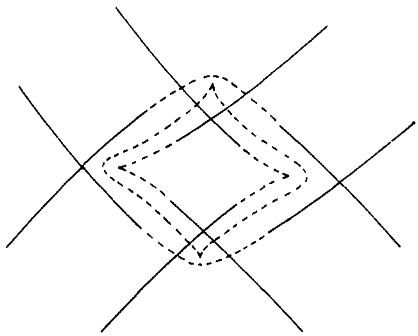

Figure 3. A (U, V) section of the ${}^0X_9$ caustic with $A = B = D = K = 0$, $C < 0$, E small and positive. The full curves indicate the A_2 and A_2^2 strata displayed explicitly in figure 1. The broken curves indicate the structure arising from unfolding the D_4^+ and A_2^3 strata (based on figures 12(*a*) and (*b*) of U82).

closely it relates to real observations of high-codimension catastrophes. We have focused our attention on a three-dimensional problem which can be visualised. In § 5 we investigate a problem where nothing is known in the general case, although again the results may easily be verified in the three-dimensional case by considering the well known geometry of these catastrophes.

5. Application II: stability of singular plane coordinate sections of cuspoid and conic umbilic catastrophes

The reader will need some acquaintance with the previous papers by Wright (1981) and Wright *et al* (1982)—henceforth referred to as I and II—to fully appreciate this section. It relates specifically to the studies presented there of the following normal forms for, respectively, the cuspoid and conic umbilic catastrophes of general codimension K:

$$\phi(s; c) = \frac{s^{K+2}}{K+2} + \sum_{n=1}^{K} c_n \frac{s^n}{n}, \tag{2}$$

$$\phi(s; c) = s_1^2 s_2 \pm \frac{s_2^K}{K} + \sum_{n=4}^{K} c_n \frac{s_2^{n-2}}{n-2} + c_3 \frac{s_1^2}{2} + c_2 s_2 + c_1 s_1. \tag{3}$$

With a slight modification of our notation above, we shall study the (i, j) sections of these normal forms, which are specified by setting all controls to zero except c_i and c_j for $K \geq i > j \geq 1$. The features occurring in these sections, and their partial stabilities, are summarised in tables 2(*a*) and (*b*) for the cuspoids and conics respectively. Recall that the symbol A_2^2 means the intersection of two A_2 strata, which is a non-local codimension-2 singularity distinct from A_2 (see II), and that the tangent space of the intersection stratum A_2^2 is the intersection of the tangent spaces of the intersecting A_2 strata.

As always, we exclude the origin because it is a completely unstable point feature. The remaining features are curve segments, straight line segments along one of the axes, or regions of the plane; this geometry was determined in detail in I and II, and is summarised in tables 2(*a*) and (*b*). The tables also show whether the singularity on a feature occurs at zero values of the state variables s. This information, plus the

Table 2. Singularity type and stability of features of singular (i, j)-sections of (a) cuspoid and (b) conic umbilic catastrophes with $K \geq i > j \geq 1$. In (b) under 'Restrictions', the symbols h, e, p stand respectively for hyperbolic, elliptic and parabolic umbilics. In both tables, under 'Features', singular planes are indicated and axial lines are shown by axis name; these singularities always occur at $s = 0$ or $s_1 = s_2 = 0$. All other features are curves, for which s or at least one of s_1 and s_2 is non-zero, as indicated.

(a)

In any section satisfying:	Singularity	Feature	Stable, or unstable to variation of:
—	A_2	$s \neq 0$: curve	Stable
K, i, j all odd	A_2^2	$s \neq 0$: curve	c_n(n even)
K, i, j all even	A_2^2	$s \neq 0$: curve	c_n(n odd)
$i \geq 3$	A_{i-1}	$s = 0$: c_i axis	$c_n(n \neq j, 1 \leq n \leq i-2)$
$j \geq 3$	A_{j-1}	$s = 0$: plane	$c_n(1 \leq n \leq j-2)$

(b)

Section	Restrictions	Singularity	Feature	Stable, or unstable to variation of:
(2, 1)	h, p	A_2	$s_1, s_2 \neq 0$	Stable
	h	A_2^2	$s_1, s_2 \neq 0$	c_3, c_n(n even ≥ 4)
(3, 1)	e, p	A_2	$s_1, s_2 \neq 0$	Stable
	—	A_{K-1}	c_3	Unstable
$(i \geq 4, 1)$	h, p	A_2	$s_1, s_2 \neq 0$	Stable
	h; i odd	A_2^2	$s_1, s_2 \neq 0$	c_3, c_n(n even ≥ 4)
	$i = 4$	A_3	c_4	c_2, c_3
	$i \geq 5$	D_{i-1}	c_i	$c_n(2 \leq n \leq i-1)$
(3, 2)	—	A_3	$s_2 \neq 0$	c_1
	—	A_{K-1}	c_3	Stable only to c_1
$(i \geq 4, 2)$	—	A_2	$s_2 \neq 0$	Stable
	h, e; i odd	A_2^2	$s_2 \neq 0$	c_n(n even ≥ 4)
	$i = 4$	A_3	c_4	c_1
	$i \geq 5$	D_{i-1}	c_i	$c_1, c_n(3 \leq n \leq i-1)$
$(i \geq 4, 3)$	—	A_3	$s_2 \neq 0$	c_1
	—	A_{K-1}	c_3	Stable only to c_1
	$i = 4$	A_3	c_4	c_1
	$i \geq 5$	D_{i-1}	c_i	$c_1, c_2, c_n(4 \leq n \leq i-2)$
	$i \geq 5$	A_{i-3}	Plane	$c_2, c_n(4 \leq n \leq i-2)$
$(i, j \geq 4)$	—	A_2	$s_2 \neq 0$	Stable
	h, e; i and j odd	A_2^2	$s_2 \neq 0$	c_n(n even ≥ 4)
	p; i and j even	A_2^2	$s_2 \neq 0$	c_2, c_n(n odd ≥ 5)
	$j \leq i-2$	D_{i-1}	c_i	$c_n(n \neq j, 1 \leq n \leq i-1)$
	$j = i-1$	D_{i-1}	c_i	$c_1, c_2, c_n(4 \leq n \leq i-2)$
	$j = 4$	A_3	Plane	c_1, c_2, c_3
	$j \geq 5$	D_{j-1}	Plane	$c_n(1 \leq n \leq j-1)$

corank and codimension of the singularity as embodied in the Arnol'd symbol for the stratum (all of which was determined in I and II), is necessary to find the local unfolding and hence the tangent space of the stratum (see appendix).

Table 2(a) displays the stability of the cuspoid sections in a very general form. As an example of its use, let us consider the (3, 1) section of the familiar canonical swallowtail catastrophe A_4, i.e. the section corresponding to the mirror plane of $\mathscr{B}$. This section has $K=3$, $i=3$, $j=1$. It was classified in I as type E2T and is similar to figure 2(*e*) of I. The following rows of table 2(*a*) apply. The first row shows that a feature of every section is a curved A_2 stratum that is completely stable. The row labelled '*K, i, j* all odd' shows that the stratum is actually A_2^2, i.e. an intersection of two A_2 strata, which is unstable to c_2, so that if c_2 is made non-zero the A_2^2 curve splits up (linearly) into two distinct A_2 curves. The row labelled '$i \geq 3$' shows that the c_3 axis is also a feature of the section, and is an A_2 stratum, which is unstable to c_n ($n \neq 1$, $1 \leq n \leq 1$), i.e. in the special case of this example it is unstable to nothing and hence is completely stable.

Table 2(*b*) for the conics needs to be more elaborate than that for the cuspoids. The general structure appears only for $i > j \geq 4$, so the low-order sections have to be treated separately, as discussed in detail in II. No distinction is made under 'singularity' between $D^{\pm}$ (generalised hyperbolic and elliptic umbilic) strata of the catastrophe because it does not affect the geometry or stability of the features in the sections we study. (This distinction is made in II.) As a simple specific illustration of the use of this table, let us apply it to the (2, 1) section of the familiar hyperbolic umbilic catastrophe D_4^+. This section of $\mathscr{B}$ is well known to display only a finite angled corner, which is also a line of self-intersection of $\mathscr{B}$, and was classified as type $B^2 1$ in II. We might ask: 'Is this self-intersection of $\mathscr{B}$ transversal?', since this is not always clear from sketches in the literature. The first two rows only of table 2(*b*) apply. The first row shows that the (2, 1) section of any (generalised) hyperbolic umbilic (D^+_{even}) displays an A_2 stratum that is completely stable, and the second row shows that in fact it is an intersection stratum A_2^2 that is never completely stable. For D_4^+ this self-intersection of $\mathscr{B}$ is unstable to c_3, so that when c_3 is varied from 0 the intersection splits up and is therefore transversal. The fact that $s_1, s_2 \neq 0$ shows that the features of (2, 1) sections of hyperbolic and parabolic umbilics do not lie along the axes (that we have used), and are curves (except for the special case of D_4^+).

As we anticipate from the general arguments in § 3, the tables show that in general the only completely stable features are curves of A_2 singularities. Apart from the origin, the only completely unstable feature is the A_{K-1} line along the c_3 axis of the (3, 1) section of every conic umbilic (with the consistent set of normal forms that we have used), as shown in table 2(*b*). For the codimension-3 conics this is a line of A_2 singularities, which are unstable despite being embedded in a two-dimensional space.

All other unstable features are partially stable. Of these, the least stable in general are the A_{K-1} lines along the c_3 axes of the (3, 2) and ($i \geq 4$, 3) conic sections, which are *stable* only to variation of c_1, and the most stable are the A_3 curve in each (3, 2) conic section and the A_3 line along the c_4 axis of each ($i \geq 4$, 2) and ($1 \geq 4$, 3) conic section, which are *unstable* only to variation of c_1.

We have considered specific applications of each table, but let us now consider curved features in general: the tables show that they are all either A_2, A_2^2 or A_3. All the A_2 *curves* are completely stable, so that the component A_2 curves of an A_2^2 curve are individually stable. The self-intersection of the A_2 strata is always stable in $\mathbb{R}^K$, but its occurrence in a plane section cannot be stable, since the A_2^2 stratum has codimension 2. Hence the A_2^2 curves are unstable, but they infold into pairs of stable A_2 curves under a generic perturbation. The A_3 strata also have codimension 2, but the A_3 curves are much more stable than the A_2^2 curves, in that they are unstable only to c_1. This difference in stability is simply an artifact of the orientation of the A_2^2 and A_3 strata in $\mathbb{R}^K$. A simple deduction from the way c_1 appears in the local unfoldings (which we have not displayed) of the A_3 curves and lines which are unstable to c_1 only is that under small perturbations they will unfold into one or three A_2 curves, but they cannot disappear completely.

We have studied these sections primarily as a way of obtaining some information in an assimilable form about the unfoldings of the whole infinite sequences of singularities A and D. However, such sections can also arise physically due to constraints, as discussed in I and II and above. For example, Pearcey and Hill (1963) computed and plotted the interference pattern near an optical focus with cylindrical symmetry, where third-order aberrations are dominant, described as optical coma. One now recognises this as the singular (2, 1) section of the swallowtail (A_4) diffraction catastrophe. More generally, if only nth-order aberration is significant, the interference pattern is the (2, 1) section of the A_{n+1} diffraction catastrophe. The caustic in such a section is, from I, an $(n+1)/n$ power-law cusp (n even) or bend (n odd). Such a feature corresponds to the top line of table 2(a), which shows that it is a completely stable A_2 feature. The effect of other degrees of aberration ceasing to be negligible is generically to further unfold the A_{n+1} singularity, i.e. to perturb the singular (2, 1) section. Our stability analysis shows that apart from the main point of focus (at the origin in canonical coordinates) the caustic is completely stable to *any* perturbation. This result is not obvious and does not follow simply from the fact that the stratum involved is A_2, because A_2 features of non-generic sections can perfectly well either disappear or split up under perturbation. In other words, a perturbation only has any effect locally in a neighbourhood of the main focus, and has no global effect on the caustic. In practice, diffraction blurring of the caustic means that it is likely to *appear* stable to small perturbations everywhere, including the main focus, in the same way that the double cusp does. One consequence of this is that the results of Pearcey and Hill will be valid (to a good approximation) more generally than appears at first sight.

6. Conclusions

In many studies involving elementary catastrophes there exists structure in the control space that is ignored in straightforward applications of elementary catastrophe theory. There are two ways to include this structure: *a priori* following Wasserman (1975), Golubitsky and Schaeffer (1979a), Damon (1983) or *a posteriori* as we have done. We have shown that a local product structure in control space is common, although its existence has rarely been acknowledged. Its effect can be to make partial unfoldings stable in practice, although they are unstable in principle, and is similar to the effect of symmetry. We have presented a way of analysing the stability of such partial unfoldings to perturbations caused by making *specific* control variations, which may be more significant in practice than the generic perturbations considered in the basic theory.

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Appendix. Examples of calculations for conic umbilic sections

We describe the calculations necessary to find the *local* form of the unfolding of $\phi(s\,; c)$, defined by equation (3) in § 5 of the text, about a point c on a stratum $\mathcal{S}^{K-m}$ of $\mathcal{B}$. These local unfoldings are necessary to find the local form of $\mathcal{B}$ for the stability analysis, which we illustrate at the end of this appendix (and also to distinguish elliptic and hyperbolic umbilic strata). In fact, all we need is the tangent space of the relevant stratum of $\mathcal{B}$, so it suffices to work to first order *locally* in the control parameters. The calculations follow the standard methods of catastrophe theory outlined by, for example, Poston and Stewart (1978) and Gilmore (1981); our aim here is to illustrate some of the essential technical details.

A1. Notation

We choose a general fixed point c_0 on $\mathcal{S}^{K-m}$ so that $\phi(s\,; c_0)$ exhibits an A_m or D_m singularity of codimension $m<K$ at some particular $s=s_0$. We introduce local variables u and τ by the substitutions $s\to s_0+u$, $c\to c_0+\tau$. We assume that we know in advance enough about the singularity about which we are expanding to know its determinacy κ, so that we know *a priori* that it suffices to work with κ-jets (i.e. to order κ) in u, and to first order in τ as explained above.

Our aim is to reduce the local unfolding of the singularity to a normal form, whose control variables are functions of τ. If the singularity has codimension m, this implies m functions $q_i(\tau)$, $1\leq i\leq m$, where $\tau\equiv(\tau_1,\ldots,\tau_K)$, such that the rank of the Jacobian $\{\partial q_i/\partial\tau_j(0)\}$ is maximal (m). Then $\mathcal{B}$ is given locally by $q(\tau)=0$, and because we have worked to only first order in τ, the $q_i(\tau)$ are all linear.

We use the following notation:

$\tilde{\phi}(u\,;\tau)$: κ-jet of local expansion of ϕ with terms independent of u omitted;

$Q(u)$: the quadratic part of the singularity $\tilde{\phi}(u\,;0)$, whose matrix is $\frac{1}{2}H$, where H is the Hessian matrix of the singularity;

$\phi_s(v\,;\tau)$: the normal form for the unfolding of the singularity, in terms of τ. For a cuspoid v is a single variable, for an umbilic $v\equiv(v_1, v_2)$;

subscripts 1 and 2 on ϕ denote partial derivatives with respect to s_1 and s_2 respectively.

Because c_0 is on $\mathcal{B}$, $\phi_1(s_0;\,c_0)=\phi_2(s_0;\,c_0)=0$, so that the linear part of $\tilde{\phi}(u\,;0)$ must be zero and can be automatically dropped.

A2. Local unfolding of an umbilic singularity

At an umbilic singularity the quadratic part $Q(u)$ of $\tilde{\phi}(u\,;0)$ must be zero. Then using determinacy, or equivalently retaining only the lowest power of u_2 whose coefficient remains finite at $\tau=0$, and shifting the u_2 origin (see § A4 below) to remove the next lowest power of u_2, gives the normal form of the local unfolding immediately.

A3. Local unfolding of a cuspoid singularity

At a cuspoid singularity $Q(u)$ is not zero, but it must be degenerate so that the determinant of the Hessian matrix H is zero. If we assume that $\phi_{11}\neq 0$ (which is always the case for the singular coordinate sections discussed in II) then we can write the matrix of Q as

$$\tfrac{1}{2}H \equiv \frac{1}{2}\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = a\begin{pmatrix} 1 & \mu \\ \mu & \mu^2 \end{pmatrix}$$

where a is a non-zero constant. If $\mu \neq 0$, the first step is to transform $\tilde{\phi}(u; \tau)$ so that $Q(u)$ is diagonal. This is necessary to find the direction in u space in which $Q(u)$ is degenerate. *Any* convenient *linear* transformation in u which diagonalises $Q(u)$ will suffice, because it will not mix different powers of u. A suitable transformation is

$$u_1 = \mu v_1 + v_2, \qquad u_2 = -v_1 + \mu v_2.$$

Then $Q(u) \equiv a(u_1^2 + 2\mu u_1 u_2 + \mu^2 u_2^2)$ becomes $Q'(v) \equiv a(1+\mu^2)^2 v_2^2$ which is clearly degenerate in v_1, and $\tilde{\phi}(u; \tau) \to \tilde{\phi}'(v; \tau)$.

We now apply a reduction algorithm, the existence of which is guaranteed by the splitting lemma and which is based on its proof (see e.g. Poston and Stewart 1978), to extract the singularity and its local unfolding. By diagonalising Q we have made the locus of Morse critical points of $\tilde{\phi}$ with respect to v_2 tangent to $(v_1; \tau)$ space at the origin of $(v; \tau)$. We find this 'Morse locus' by solving

$$\partial\tilde{\phi}'(v; \tau)/\partial v_2 = 0 \qquad \text{for } v_2 \text{ as a function of } v_1 \text{ and } \tau.$$

Then the restriction of $\tilde{\phi}'(v; \tau)$ to the Morse locus, namely $\phi_s(v_1; \tau) \equiv \tilde{\phi}'(v_1, v_2(v_1; \tau); \tau)$, is the local unfolding of the singularity.

In finding $v_2(v_1; \tau)$ we remember that we are working only to order κ in v_1 and to linear order in τ. We only need find v_2 to sufficient accuracy to give ϕ_s to the correct order, which makes the problem much simpler than it appears. In fact, it often amounts simply to setting $v_2 = 0$.

A4. Shift lemma

The final step in reducing the local unfolding to a normal form is to translate the variable (v_1) to remove the 'penultimate term', which we define to be the highest power but one, using the following result.

If $f = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_{n-1}x$ and $c_r = O(\tau)$, $1 \leqslant r \leqslant n-1$, then

$$f = x'^n + c_2 x'^{n-2} + \ldots + c_{n-1}x' + O(\tau^2) \qquad \text{where } x' = x + c_1/n. \tag{A1}$$

This shows that we can simply drop the penultimate term, although for an umbilic singularity we must also transform the $u_1^2 u_2$ term using (A1).

A5. Example: local unfolding about a singular coordinate section

To illustrate the method we will find a local unfolding about a singularity at a point in an $(i \geqslant 2, 1)$ plane (i.e. with all c_n other than c_i and c_1 set to zero) which has $s_1, s_2 \neq 0$ (see II). In this case the Hessian is not diagonal. Information derived as in II and quoted in table 2(b) shows that this singularity should be A_2, which is 3-determined. Expanding to $O(u^3)$ and $O(\tau)$ about the singularity and dropping terms linear in u

which are independent of τ, as explained in § A1, gives

$$\tilde{\phi}(u;\tau)=\left[u_1^2u_2+\left(a_K+\sum_{n=4}^{K}(c_n+\tau_n)a_{n-2}\right)u_2^3\right]$$
$$+\left[[s_2+\tfrac{1}{2}(c_3+\tau_3)]u_1^2+2s_1u_1u_2+\left(\tfrac{1}{2}(K-1)s_2^{K-2}+\sum_{n=4}^{K}(c_n+\tau_n)b_n\right)u_2^2\right]$$
$$+\left(\sum_{n=4}^{K}\tau_n s_2^{n-3}u_2+\tau_3 s_1u_1+\tau_2u_2+\tau_1u_1\right),$$

where

$$a_n\equiv\tfrac{1}{6}(n-1)(n-2)s_2^{n-3},\qquad b_n\equiv\tfrac{1}{2}(n-3)s_2^{n-4},$$
$$c_0\equiv(c_1,c_2,\ldots,c_K),\qquad s_0\equiv(s_1,s_2),$$

and all c_n, $n\geqslant 3$, are zero except for $n=i$.

The quadratic part with $\tau=0$ must have the form

$$Q(u)\equiv(s_2+\tfrac{1}{2}c_3)(u_1^2+2\mu u_1u_2+\mu^2u_2^2).$$

(This could, of course, be shown explicitly using the relations between c_i, c_j, s_1 and s_2 found as in II.) Diagonalising as explained in § A3, $\phi(u;\tau)$ becomes

$$\tilde{\phi}'(v;\tau)=[(\mu+A)v_2^3+(2\mu^2-1-3\mu^2A)v_1v_2^2+\mu(\mu^2-2+3A)v_1^2v_2-(\mu^2+A)v_1^3]$$
$$+(s_2+\tfrac{1}{2}c_3)(1+\mu^2)^2v_2^2+\left(\sum_{n=4}^{K}\tau_n s_2^{n-3}+\tau_2\right)(\mu v_2-v_1)+(\tau_3s_1+\tau_1)(\mu v_1+v_2)$$

where

$$A=a_K+\sum_{n=4}^{K}(c_n+\tau_n)a_{n-2}.$$

The 'Morse locus' is given by

$$\partial\tilde{\phi}'(v;\tau)/\partial v_2\equiv 3(\mu+A)v_2^2+2(2\mu^2-1-3\mu^2A)v_1v_2+\mu(\mu^2-2+3A)v_1^2$$
$$+2(s_2+\tfrac{1}{2}c_3)(1+\mu^2)v_2+\mu\left(\sum_{n=4}^{K}\tau_n s_2^{n-3}+\tau_2\right)+\tau_3s_1+\tau_1=0.$$

Extracting the term linear in v_2, we have $v_2=O(\tau)+O(v_1^2)+O(v_1v_2)+O(v_2^2)$. Iterating to $O(\tau)$ and $O(v_1^3)$, this becomes

$$v_2=O(\tau)+O(v_1^2)+O(\tau v_1),\qquad v_2^2=O(\tau v_1^2),\qquad v_2^3=0.$$

Substituting into $\tilde{\phi}'(v;\tau)$, retaining only terms to $O(v_1^3)$ and only lowest order in τ in all coefficients, we find

$$\phi_s(v_1;\tau)\equiv\tilde{\phi}'(v_1,v_2(v_1;\tau);\tau)$$
$$=-(\mu^2+a_K[+c_ib_i\text{ if }i\geqslant 4])v_1^3+O(\tau)v_1^2-\left(\sum_{n=4}^{K}\tau_n s_2^{n-3}+\tau_2\right)v_1$$
$$+\mu(\tau_3s_1+\tau_1)v_1.$$

By the shift lemma, we may drop the term in v_1^2, and the result is identical to setting $v_2=0$ in $\tilde{\phi}'(v;\tau)$. Checking that the coefficient of v_1^3 is not zero confirms the A_2 singularity.

A6. Example: tangent space and partial stability

The tangent space of any stratum of codimension m is given by the simultaneous solution of m (linear) equations. The tangent space of (the A_2 stratum of) $\mathscr{B}$ at the point considered in § A5 has equation

$$\mu\tau_1 - \tau_2 + \mu s_1\tau_3 - \sum_{n=4}^{K} s_2^{n-3}\tau_n = 0. \tag{A2}$$

Since this equation contains variations in all directions in control space, all points of this particular feature (a curve of A_2 singularities) are completely stable, as stated in table 2(*b*).

In the $h(2, 1)$ and $h(i \text{ odd} \geqslant 5, 1)$ sections, these curves are actually self-intersections of the A_2 stratum. On the 'other sheet', s_1 and s_2 have opposite sign, but in these sections $\mu = s_1/s_2$ and so does not change sign. Therefore, in terms of s-values on one sheet, the equation of the tangent space of the other sheet is

$$\mu\tau_1 - \tau_2 - \mu s_1\tau_3 - \sum_{n=4}^{K} (-s_2)^{n-3}\tau_n = 0. \tag{A3}$$

The tangent space of the intersection stratum is given by the simultaneous solution of (A2) and (A3). Adding and subtracting these equations leads to two equations relating disjoint sets of τ_n, from which the unstable controls may easily be picked out, and are as stated in table 2(*b*).

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